

VII. *On Centripetal Forces.* By Edward Waring, M. D.
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P R O P. I.

I. **L**ET a curve PpN (Tab. II. fig. 1.), of which the perpendiculars to the two nearest points P and p of the curve are PO and pO , and consequently O the center of a circle, which has the same curvature as the given curve in the point P ; draw PY and ly tangents to the curve in the points P and p ; from S draw Sy and SbY respectively perpendiculars to the tangents ly and PY ; and let SbY cut the tangent ly in b ; then will ultimately bY ($-P$) be the decrement of the perpendicular $SY = P$; and the triangles lbY and POp be similar: for the angles POp and bly are equal, and the angles lYb and OPp right ones; therefore $PO : Pp :: lY$ ultimately $= PY : Yb$ decrement of the perpendicular, whence $Pp = \frac{Yb \times PO}{lY} = \frac{Yb \times PO}{PY}$.

1.2. Fig. 2. and 1. The force in the direction PS is as the ultimate ratio of $2 \times QR$ (the space through which a body is drawn from the direction of its motion in the tangent in a given time towards the center of force); but ultimately $2QR = \frac{2QP^2}{PV}$, where QP is as the space described in a given time, and consequently as the velocity (V) of the body at the given point P , and PV the chord of curvature in the direction SP .

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1.3. The increment (Pp) of the space divided by the velocity V is ultimately as the increment of the time, and = the increment of the velocity (\dot{V}) divided by the force $\frac{2V^2}{PV} \times \frac{PY}{SP}$ in the direction of the tangent, that is, $\frac{Pp}{V} = \frac{\dot{V} \times PV \times SP}{2V^2 \times PY}$; for Pp substitute $\frac{Yb \times PO}{PY}$, and there results $\frac{-\dot{P} \times PO}{PY \times V} = \frac{\dot{V} \times PV \times SP}{2V^2 \times PY}$; and consequently $\frac{-2\dot{P} \times PO}{SP \times PV} = \frac{\dot{V}}{V}$; but $\frac{SP \times PV}{2PO} = SY = P$, whence $\frac{-\dot{P}}{P} = \frac{\dot{V}}{V}$, and $V = \frac{a}{P}$, where a is an invariable quantity.

Cor. Since $V \times P$, that is, SY the perpendicular multiplied into the velocity (which is ultimately as Pp the space described in a given time) is ultimately as the area described round the center S in a given time; but this rectangle = a , a given quantity; therefore the area, described round the center of force S in a given time, will be a given quantity, and thence in unequal times will be proportional to the times.

1.4. The sagitta QR is ultimately as the force, when the time is given; and when the time is not given, it will be as the force into the square of the time; from which expression, by substituting for QR and the time their values, may be deduced several others.

Sir ISAAC NEWTON has demonstrated this proposition with the greatest simplicity; and this is given to shew, that the same proposition may be deduced from different principles.

P R O P. II.

1. Fig. 3. Given the relation between SP' the distance from a point S , and SY' a perpendicular from the point S to $P'Y$, a line touching a curve in the point P' ; to find the relation
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between Sp' and Sy' (a perpendicular from the point S to $p'y$, a line touching the curve in the point p'); in which two curves $PP'L$ and $pp'l$, the forces and velocities at any equal distances SP and Sp' are equal, and consequently the perpendiculars SY and Sy , at the above-mentioned equal distances SP and Sp' are to each other in a given ratio $N : n$.

In the equation expressing the relation between SP' and SY' for SP' and SY' write respectively Sp' and $\frac{Sy' \times N}{n}$, and there results the equation sought: for the distances SP and Sp' being equal, the perpendiculars SY' and Sy' are as $N : n$.

Ex. 1. Let S be the focus of a conic section, then will $\frac{1}{4} C^2 \times \frac{D}{T \pm D} = SY^2 = P^2$, where T and C denote its transverse and conjugate axes, and D the distance SP ; for P write $\frac{N}{n} \times p$, and there results the equation $\frac{1}{4} C^2 \times \frac{D}{T \pm D} = \frac{N^2}{n^2} \times p^2$, which is an equation to a conic section of the same name (*viz.* ellipse, parabola, or hyperbola) as the given curve, of which the transverse axis is T , and conjugate $= \frac{C \times n}{N}$, and perpendicular from the focus to the tangent $= p$. If T and C are infinite, and consequently the curve a parabola, and the equation $\frac{1}{4} L \times D = P^2$, then will the *latus rectum* of the resulting equation be $\frac{L \times n^2}{N^2}$.

Ex. 2. Let S be the center of the logarithmic spiral, then will the equation be $a \times SP = a \times D = SY = P$, and consequently the resulting equation $a \times D = \frac{N}{n} \times p$, whence $\frac{an}{N} \times D = p$ an equation to a logarithmic spiral having the same center.

Ex. 3. Let T and C be the semi-conjugate axes of a conic section, and S its center; then will the equation expressing the

the relation between the distance D and perpendicular P be $D^2 \pm \frac{T^2 C^2}{P^2} = T^2 \pm C^2$; for P write as before $\frac{Np}{n}$, and there results the equation $D^2 \pm \frac{n^2 T^2 C^2}{N^2 p^2} = T^2 \pm C^2$, an equation to a conic section of the same name, of which the transverse and conjugate diameters are respectively two roots (x) of the equation $x^2 \pm \frac{n^2 T^2 C^2}{N^2 x^2} = T^2 \pm C^2$, because in this case $p = D$.

The sum or difference of the squares of the transverse and conjugate diameters, in all the resulting equations, will be the same.

Cor. In every equal distance, the chord of curvature passing through the center of force is the same; for the forces in that direction, and the velocities at every equal altitude are the same.

P R O P. III.

1. Fig. 4. and 3. Given an equation $A = o$, expressing the relation between the absciss $SM = x$ and ordinate $MP = y$; to find the equation expressing the relation between $SP = \sqrt{x^2 + y^2}$ and $SY = P$, the perpendicular from S on the tangent PY . From the equation $A = o$ find $\dot{x} = B\dot{y}$, which substitute for \dot{x} in the equation $(\dot{x}^2 + \dot{y}^2)^{\frac{1}{2}} \times P = \dot{x}y \pm xy$ deduced from the similar triangles $Pl o$, MTP , and STY , where $lo = \dot{x}$ and $Po = \dot{y}$; let the resulting equation be $C = o$; reduce the three equations $A = o$, $C = o$, and $x^2 + y^2 = SP^2 = D^2$ into one, so that the unknown quantities x and y may be exterminated, and there results an equation expressing the relation between D and P .

Cor. Hence from the equation expressing the relation between x and y , the absciss and ordinate of a curve, can be deduced an equation expressing the relation between the distance SP and perpendicular SY ; and from the equation expressing

the relation between the distance SP and SY can be deduced an equation expressing the relation between the distance Sp and perpendicular Sy from the point S to the tangent py of a curve, whose force and velocity at every equal distance is the same as in the given curve, but the direction different.

2. Given an equation $K=0$ expressing the relation between $SP=D$ and $SY=P$; to find an equation expressing the relation between $SM=x$ and $PM=y$, the absciss and ordinate of the same curve.

In the given equation $K=0$ for D and P write respectively $\sqrt{x^2+y^2}$ and $\frac{y\dot{x}\pm x\dot{y}}{\sqrt{y^2+x^2}}$, and there results a fluxional equation $L=0$ of the first order, of which the fluent expresses the general relation between x and y .

Cor. If in the given equation for P be wrote nP' , there results the equation $K'=0$, which expresses the relation between the perpendicular $Sy=P'$ and distance $Sp=D'$ of every curve, which at equal distances has the same velocity and force tending to S; reduce the equations $K'=0$, $D=\sqrt{x^2+y^2}$ and $nP'=\frac{y\dot{x}\pm x\dot{y}}{\sqrt{x^2+y^2}}$ into one, so that D and P' may be exterminated, and there will result the same fluxional equation of the first order, expressing the relation between x , y , and their fluxions, whatever may be the value of n . The general fluent of this fluxional equation contains the relation between the absciss and ordinates of all curves, which have the same force and velocity at the same distance as the force and velocity in the given curve.

P R O P. IV.

1. Let a body move in a given curve PH (fig. 5.), of which the velocity (v) at any point P is given: and let the forces f'' , f''' , &c. tending to all the given centers S'' , S''' , &c. (except two S and S') be given; to find the forces f and f' tending to the two points S and S'.

Draw a line PO perpendicular to the tangent yPy' ; and from the given centers S, S', S'', &c. draw lines $S'l$ and Sy , $S'l'$ and $S'y'$, $S''l''$ and $S''y''$, &c. perpendicular to the lines PO and yPy' , &c.; then will $\frac{v^2}{PO} = f \times \frac{Pl}{PS} \pm f' \times \frac{Pl'}{PS'} \pm f'' \times \frac{Pl''}{PS''} \pm \&c.$ (where PO is the radius of the circle having the same curvature as the curve in the point P), and $\frac{-vv'}{A} = f \times \frac{Py}{PS} \pm f' \times \frac{Py'}{PS'} \pm \&c.$ (where A denotes the arc of the curve PH); from the *data* may be deduced all the quantities contained in the above mentioned two equations, except f and f' ; and consequently from the two given simple equations be deduced the forces sought f and f' .

2. Let the velocity of the body moving in the given curve be supposed always uniform; then $f \times \frac{Py}{PS} \pm f' \times \frac{Py'}{PS'} \pm f'' \times \frac{Py''}{PS''} \pm \&c. = 0.$

Ex. Let the curve HPl be an ellipse, and the two foci S and S' the centers of forces; then will $f \times \frac{Py}{SP} = f' \times \frac{Py'}{S'P}$, but the angle $SPy = S'Py'$, and consequently $\frac{Py}{SP} = \frac{Py'}{S'P}$ and $f = f'$; but since $\frac{v^2}{PO} = f \times \frac{Sy}{SP} + f \times \frac{Sy'}{S'P} = 2f \times \frac{Sy}{SP}$, and $v = a$, then will $f = \frac{a^2 \times SP}{2Sy \times PO}$ be the force tending to each focus.

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In these and the subsequent cases the lines Py , Py' , Py'' , &c. are to be taken negatively or affirmatively, as they are situated on the same or different sides of P ; and in the same manner the lines Pl , Pl' , Pl'' , &c. are to be taken negatively or affirmatively as they are situated on the same or different sides of the tangent yPy' , &c.

3. Let the centers M , M' , M'' , M''' , &c. of forces be points not situated in the plane of the given curve HPI , &c. and the forces f''' , f'''' , &c. tending to each of the centers M''' , M'''' , &c. (except three M , M' , and M'') be given; to find the forces f , f' , and f'' tending to those three points M , M' , and M'' .

Draw MS , $M'S'$, $M''S''$, &c. perpendicular to the plane HPI , &c. from the above-mentioned points, and assume the equation $f \times \frac{MS}{\sqrt{MS^2 + SP^2}} \pm f' \times \frac{M'S'}{\sqrt{M'S'^2 + S'P^2}} \pm f'' \times \frac{M''S''}{\sqrt{M''S''^2 + S''P^2}} \pm f''' \times \frac{M'''S'''}{\sqrt{M'''S'''^2 + S'''P^2}} + \&c. = 0$, and the two preceding equations $\frac{v^2}{PO} = f \times \frac{Pl}{PM} \pm f' \times \frac{Pl'}{PM'} \pm f'' \times \frac{Pl''}{PM''} \pm \&c.$ and $\frac{-v\dot{\phi}}{A} = f \times \frac{Py}{PM} \pm f' \times \frac{Py'}{PM'} \pm f'' \times \frac{Py''}{PM''} \pm \&c.$; from the *data* may be found all the quantities f''' , f'''' , &c.; and consequently from the above mentioned equations may be deduced the forces f , f' , and f'' .

4. Let the body move in different planes, that is, in a curve of double curvature at the same points; draw PR a tangent to the curve at the point P , and PQ an arc of the curve of double curvature; draw also two planes PRV and PRT , cutting one another in the line PR ; from the point Q let fall QV and QT perpendicular to those planes respectively, and from the points V and T draw Vv and Tt respectively perpendicular to the line PR ; let v be the velocity of a body moving in the given

curve at the point P, and assume $\frac{PQ^2}{Vv} = 2C$ and $\frac{PQ^2}{Tt} = 2C'$ respectively; from the given centers of forces M, M', M'', M''', &c. draw MS, MS', M''S'', M''''S''', &c.; Ms, M's', M''s'', M''''s''', &c. respectively perpendicular to the two planes RPV and RPT; and PL and P'l perpendicular to the line PR in the same two planes RPV and RPT; and also SP, S'P, S''P, S''''P, &c.; sP, s'P, s''P, &c.: from the points S, S', S'', S''', &c. s, s', s'', &c. draw the lines SH, S'H', S''H'', S''''H''', &c. sb, s'b', s''b'', s''''b''', &c. respectively perpendicular to the lines PL and P'l; and SK, S'K', S''K'', S''''K''', &c. sk, s'k', s''k'', s''''k''', &c. perpendicular to the line RP; and let the forces f'''' , f'''''' , &c. tending to all the points M''', M''''', &c. (except three, M, M', and M'') be given; then from the three given equations $\frac{v^2}{C} = \frac{PH}{MP} \times f \pm \frac{PH'}{M'P} \times f' \pm \frac{PH''}{M''P} \times f'' \pm \&c.$ and $\frac{v^2}{C'}$
 $= \frac{Pb}{MP} \times f \pm \frac{Pb'}{M'P} \times f' \pm \frac{Pb''}{M''P} \times f'' \pm \&c.$ and $\frac{-v\dot{v}}{A} = \frac{PK}{MP} \times f \pm \frac{PK'}{M'P} \times$
 $f' \pm \frac{PK''}{M''P} \times f'' \pm \frac{PK'''}{M''''P} \times f'''' \pm \&c. = \frac{Pk}{MP} \times f \pm \frac{Pk'}{M'P} \times f' \pm \frac{Pk''}{M''P} \times f''$
 $\pm \&c.$ which contain only three unknown quantities, can be deduced the forces f , f' , and f'' , required, tending to the points M, M', and M''.

P R O P. V.

Let a body acted on by forces tending to any given points S, S', S'', &c. move in a given curve, to find its velocity in any point of the curve.

Find the fluent of the fluxion ($f \times \frac{Py}{PS} \pm f' \times \frac{Py'}{PS'} \pm f'' \times \frac{Py''}{PS''} \pm f''' \times \frac{Py'''}{PS'''} \pm \&c.$) $\dot{A} = f\dot{D} \pm f'\dot{D}' \pm f''\dot{D}'' \pm \&c. = -v\dot{v}$ when the forces are all contained in the same plane; or the fluent of $(f \times$

$(f \times \frac{Py}{PM} \pm f' \times \frac{Py'}{PM'} \pm f'' \times \frac{Py''}{PM''} \pm \&c.) \times \dot{A}$ (when contained in different planes) $= f \times \dot{PM} \pm f' \times \dot{PM}' \pm f'' \times \dot{PM}'' \pm \&c. = f \times \dot{D} \pm f' \times \dot{D}' \pm f'' \times \dot{D}'' \pm \&c.$; but since $f, f', f'', \&c.$ are given functions of the quantities $D, D', D'', \&c.$ the fluents of $f \times \dot{D}, f' \times \dot{D}', f'' \times \dot{D}'', \&c.$ can be found; which, when properly corrected will be as $\frac{v^2}{2} = \frac{1}{2}$ the square of the velocity in any point P. A denotes the arc of the curve, and $D, D', D'', \&c.$ the respective distances of the body from the centers of forces.

Cor. The increment of the time of describing any arc of the above-mentioned curve will be as the increment of the arc $= \dot{A}$ divided by the velocity found above, and consequently the time itself will be as the fluent of it properly corrected.

P R O P. VI.

1. Let a body move in any curve, and be acted on by forces tending to any given points, $S, S', S'', S''', \&c.$; all of which, except the force f tending to the point S , let be given, to find f the force tending to S .

Let $Sy, S'y', S''y'', \&c.$ be perpendicular to the tangent Py of the curve at the point P ; resolve the forces $f, f', f'', f''', \&c.$ tending to $S, S', S'', S''', \&c.$ respectively into two forces, of which one acts perpendicular to Py , the other, $S'l, S'l', S''l'', \&c.$ perpendicular to PO , which is perpendicular to Py ; let PO be radius of the circle of the same curvature as the curve, and v the velocity of the body at the point P ; then will $\frac{v^2}{PO} = f \times \frac{Sy}{SP} \pm f' \times \frac{S'y'}{S'P} \pm f'' \times \frac{S''y''}{S''P} \pm f''' \times \frac{S'''y'''}{S'''P} \pm \&c.$ and -

$$\frac{v\dot{v}}{A} = f \times \frac{Py}{SP} \pm f' \times \frac{Py'}{S'P} \pm f'' \times \frac{Py''}{S''P} \pm f''' \times \frac{Py'''}{S'''P} \pm \&c. : \text{ for } \frac{Sy \times PO}{SP} =$$

$\frac{1}{2}$ chord of the circle of curvature, which passes through S, write C; and for PO $\times (\pm f' \times \frac{S'y'}{S'P} \pm f'' \times \frac{S''y''}{S''P} \pm \&c.)$ substitute H, and for $\frac{Py}{SP}$ write B; and for $\pm f' \times \frac{Py'}{S'P} \pm f'' \times \frac{Py''}{S''P} + \&c.$ substitute D, and the two preceding equations become $v^2 = f \times C + H$ and $-v\dot{v} = (Bf + D)\dot{A}$, where \dot{A} denotes as before the increment of the arc of the curve: from the first equation $v\dot{v} = \frac{f \times \dot{C} + C\dot{f} + \dot{H}}{2} = -(Bf\dot{A} + D\dot{A})$ and consequently $C\dot{f} + (\dot{C} + 2B\dot{A})f + \dot{H} + 2D\dot{A} = 0$, from which fluxional equation may be deduced the force f tending to the center (S) $= -C^{-1} \times e^{-\int \frac{2B\dot{A}}{C}} \times \int (\dot{H} + 2D\dot{A}) \times e^{\int \frac{2B\dot{A}}{C}}$, where e is the number, whose hyper. log. = 1.

Cor. Fig. 1. Let $f', f'', f''', \&c.$ be each = 0, then will $D = 0$, $H = 0$, and consequently $\int (2\dot{D}A + \dot{H}) \times e^{\int \frac{2B\dot{A}}{C}} = \text{const} = a$, and $f = -a \times C^{-1} \times e^{-\int \frac{2B\dot{A}}{C}} = \frac{2B \times S_j \times PO}{C \times P_j} = \frac{2S_j}{S_j} = -aC^{-1} \times e^{-\int \frac{2S_j}{S_j}}$; whence $f = \frac{-a}{S_j^2 \times C}$ as is generally known, where a denotes an invariable quantity.

Cor. The force f being found, the square of the velocity may be deduced from the equation $v^2 = f \times C + H$, and the time from the fluent of the fluxion $\frac{\dot{A}}{v} = \frac{\dot{A}}{\sqrt{f \times C + H}}$.

2. Let the body move in a curve of double curvature, and let the forces $f'', f''', \&c.$ tending to all the points $M'', M''', \&c.$ (except two, M and M') be given; to find the forces tending to the points M and M'.

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Assume the three equations before given in Prop. 4. viz.
 $\frac{v^2}{C} = \frac{PH}{MP} \times f \pm \frac{PH'}{M'P} \times f' \pm \frac{PH''}{M''P} \times f'' \pm \&c.$ $\frac{v^2}{C'} = \frac{Pb}{MP} \times f \pm \frac{Pb'}{M'P} \times f' \pm \frac{Pb''}{M''P} \times f'' \pm \&c.$ and $-v\dot{v} = \left(\frac{PK}{MP} \times f \pm \frac{PK'}{M'P} \times f' \pm \frac{PK''}{M''P} \times f'' \pm \&c. = \frac{Pk}{MP} \times f \pm \frac{Pk'}{M'P} \times f' \pm \&c. \right) \times \dot{A}$, from the two former may be deduced the equations $v\dot{v} = \alpha\dot{f} + \beta\dot{f}' + f\dot{\alpha} + f'\dot{\beta} + \dot{\gamma}$, and $v\dot{v} = \alpha'\dot{f} + \beta'\dot{f}' + f\dot{\alpha}' + f'\dot{\beta}' + \dot{\gamma}'$, where $\alpha = \frac{C \times PH}{2MP}$, $\beta = \pm \frac{C \times PH'}{2M'P}$, $\gamma = \pm \frac{1}{2} C \left(\frac{PH'' \times f''}{M''P} \pm \frac{PH''' \times f'''}{M'''P} \pm \&c. \right)$; $\alpha' = \frac{C' \times Pb}{2MP}$, $\beta' = \pm \frac{C' \times Pb'}{2M'P}$, $\gamma' = \pm \frac{1}{2} C' \left(\frac{Pb'' \times f''}{M''P} \pm \frac{Pb''' \times f'''}{M'''P} \pm \&c. \right)$, whence may be derived the two equations $\alpha\dot{f} + \beta\dot{f}' + f\dot{\alpha} + f'\dot{\beta} + \dot{\gamma} = \alpha'\dot{f} + \beta'\dot{f}' + f\dot{\alpha}' + f'\dot{\beta}' + \dot{\gamma}' = \pi f \pm \rho f' \pm \sigma$, where $\pi = - \left(\frac{PK}{MP} = \frac{Pk}{MP} \right) \times \dot{A}$, $\rho = \pm \left(\frac{PK'}{M'P} = \frac{Pk'}{M'P} \right) \times \dot{A}$, $\sigma = - \left(\pm \frac{PK''}{M''P} \times f'' \pm \frac{PK'''}{M'''P} \times f''' \pm \&c. = \&c. \right) \times \dot{A}$.

Reduce these two equations to one, so that f' , f'' , &c. and their fluxions, may be exterminated, and there results a fluxional equation of the formula $H\dot{f} + Kf + Lf + M = 0$, where H, K, L, and M, are functions of one of the before-mentioned variable quantities (for example, $MP = W$) which may be supposed to flow uniformly, and its fluxion.

P R O P. VII.

1. Fig. 6. Given the force tending to any point S, the velocity and direction of the body; to find the curve described.

Let the body acted on by a force f tending to S, at the distance D' from S be projected in the direction $P'Y'$, with a

velocity H ; and let the perpendicular from S to the tangent $P'Y'$ be A ; from the general fluent of $f \times \dot{D}$, where D denotes the distance from S , and f is a function of D , properly corrected find its velocity V at distance D , and consequently the perpendicular SY from the center S to the tangent PY at distance $D=SP$, which will be $\frac{A \times H}{V} = SY$; but A and H are given quantities, and V a known function of D ; therefore SY and $\sqrt{SP^2(D^2) - SY^2} = PY$ will be known functions of D ; and from the similar triangles SPY and PQT may be deduced $PY : SY :: PT = \dot{D} : QT$, and consequently $SP \times QT = D \times \frac{\dot{D} \times SY}{PY}$ (which is a known function of D multiplied into \dot{D}) will be as the increment of the area described round the center of force, of which the fluent properly corrected is proportional to the area described round the center of force, and consequently to the time. In like manner, $\frac{Sy \times \dot{D}}{D \times PY} = \frac{QT}{SP}$ (proportional to the increment of the angle described by the body round S) is a function of D multiplied into \dot{D} , of which the fluent properly corrected, or angle, will be as a function of D .

1.2. Fig. 7. Given the above-mentioned force, &c.; to find an equation expressing the relation between the absciss $SM = x$ and ordinate $MP = y$ of the curve described, and their fluxions.

From the similar triangles Ppo and LPM can be deduced $po = y : oP = \dot{x} :: PM = y : LM = \frac{y\dot{x}}{y}$; but $LM \pm SM = \frac{y\dot{x}}{y} \pm x = \frac{y\dot{x} \pm x\dot{y}}{y} = LS$; and consequently $Pp = \sqrt{x^2 + y^2} : po = y :: LS = \frac{y\dot{x} \pm x\dot{y}}{y} : SY = \frac{y\dot{x} \pm x\dot{y}}{\sqrt{x^2 + y^2}}$; but SY is a function to be deduced

as above of $SP = \sqrt{x^2 + y^2}$, whence the fluxional equation

$$\frac{y\dot{x} \pm x\dot{y}}{\sqrt{x^2 + y^2}} = \phi : (x^2 + y^2).$$

2. Fig. 8. Let a body be acted on by any number of forces ($f, f', f'', f''', \&c.$) in the same plane tending to the given points $S, S', S'', S''', \&c.$; to find an equation expressing the relation between $SP = D$ and $S'P = D'$, and their fluxions, where P is a point situated in the curve which the body describes.

Suppose YP a tangent to the curve at the point P , and PZ perpendicular to it; and resolve all the forces tending to $S, S', S'', \&c.$ respectively into two others; one in the direction PY , and the other in the direction PZ ; substitute for $SP, S'P, S''P, S'''P, \&c.$ respectively $D, D', D'', D''', \&c.$; and suppose $SY, S'Y', S''Y'', S'''Y''', \&c.$ perpendicular to the line PY ; then will the triangles PQT and $SPY, PQ'T'$ and $S'PY'$ be similar, where PQ denotes a very small arc, and QT and QT' are perpendicular to the lines SP and $S'P$; hence $PQ = \frac{PT \times SP}{PY} = \frac{\dot{D} \times D}{PY} = \frac{PT' \times S'P}{PY'} = \frac{\dot{D}' \times D'}{PY'}$; and consequently $PY : PY' :: \dot{D} \times D : \dot{D}' \times D'$; and if the quantities D, D', \dot{D} and \dot{D}' are given, the ratio of $PY : PY'$ will be given; which being given, together with the line $SS' = a$, the lines PY and PY', SY and $S'Y'$, can be found; for, drawing SL parallel to PY , and meeting $S'Y'$ in L , let $PY' = m \times PY$, then $YY' = (m \pm 1) PY = SL, SY = \sqrt{SP^2 - PY^2} = \sqrt{D^2 - PY^2}, S'Y' = \sqrt{S'P^2 - PY'^2} = \sqrt{D'^2 - m^2 \times PY^2}, LS' = S'Y' \pm SY = \pm \sqrt{D'^2 - m^2 PY^2} \pm \sqrt{D^2 - PY^2}$; and $SS'^2 = SL^2 + LS'^2$ an equation in which all quantities (except PY) are given, and consequently PY is determined by an equation, which will be a quadratic; but PY being found, from thence PY', SY and $S'Y'$

$S'Y'$ may be deduced, which are consequently all functions of D, D', \dot{D}, \dot{D}' , and invariable quantities; and their fluxions $P\dot{Y}'$, $S\dot{Y}'$, and $S'\dot{Y}'$ functions of D, D', \dot{D}, \dot{D}' , \ddot{D} , and \ddot{D}' : from the similar triangles before given $S\dot{Y} : PQ = \frac{D\dot{D}}{PY} :: PY : \frac{D\dot{D}}{S\dot{Y}} = PO$ the radius of curvature

hence PO is a function of D, D', \dot{D}, \dot{D}' , and \ddot{D}' , if $\ddot{D} = 0$;

and from $D, D', SS', \dot{D}, \dot{D}'$, and the point S'' given in position can be determined $S''P, S''Y''$ and PY'' ; for let $S''b = C$ be drawn perpendicular to $SS' = a$, and $Sb = b$; then will $S'l$ (if Pl be a perpendicular from the point P to the line SS') $= \pm \frac{a^2 + D^2 - D'^2}{2a}$, and $S'l = \pm \frac{a^2 + D'^2 - D^2}{2a}$, and $Pl = \sqrt{(SP^2 - S'l^2)}$, and

$S''P = \sqrt{((b \pm S'l)^2 + (C \pm Pl)^2)}$; draw $S''Y''$ perpendicular to the tangent PY , and cutting the lines SS' and SK parallel to

PY in o and n respectively; then will $ob = \frac{C \times \sqrt{(SS'^2 - (PY \pm PY'')^2)}}{(PY \pm PY'') = l}$,

$S''o = \frac{C \times 6S''}{l}$; (and from the similar triangles $S''ob$ and $So n$) $on =$

$(b \pm ob) \times \frac{ob}{S''o}$; whence $S''Y'' = \pm S''o \pm on \pm SY$ will be a

known function of D, D', \dot{D} and \dot{D}' , and invariable quantities: the same may be predicated of similar lines drawn to the

centers S''' , S'''' , &c.; and consequently $(f \times \frac{PY}{SP} \pm f' \times \frac{PY'}{S'P}$

$\pm f'' \times \frac{PY''}{S''P} \pm f''' \times \frac{PY'''}{S'''P} \pm \&c.) \times \dot{A}$ (where \dot{A} , as before, denotes the fluxion of the arc of the curve) $= f \times \dot{D} \pm f' \times \dot{D}' \pm$

$f'' \times \dot{D}'' \pm f''' \times \dot{D}''' \pm \&c. = -v\dot{v}$, if v denotes the velocity; but as f, f', f'', f''' , &c. are functions of D, D', D'', D''' , &c. respectively, the fluent of the above mentioned quantity

$f\dot{D} \pm f'\dot{D}' \pm f''\dot{D}'' \pm \&c.$ can be found in terms of D, D', D'' ,

D'' , D''' , &c. from the fluents of the fluxions $f\dot{D}$, $f'\dot{D}'$, &c.; and consequently in terms of D and D' , which let be Z , then will $Z = \frac{-v^2}{2}$; but $v^2 = -2Z = PO \times (f \times \frac{SY}{SP} \pm f' \times \frac{S'Y'}{S'P} \pm f'' \times \frac{S''Y''}{S''P} \pm f''' \times \frac{S'''Y'''}{S'''P} \pm \&c.)$ a fluxional equation of the second order expressing the relation between D and D' , and their fluxions.

2. To find an equation expressing the relation between $x=SM$ and $y=MP$, where SM (x) is the absciss beginning from S and continued in the line SS' , and MP (y) the perpendicular ordinate of the curve described by a body acted on by the above mentioned forces: in the fluxional equation found before for D and D' and their fluxions substitute $(x^2 + y^2)^{\frac{1}{2}}$ and $((SS' \pm x)^2 + y^2)^{\frac{1}{2}}$ and their fluxions, and there results the equation sought.

Cor. It easily appears, that the general fluent may contain two invariable quantities to be assumed at will, or according to the conditions of the problem; that is, at a given distance the velocity and the direction may be assumed at will, and consequently the general fluxional equation expressing the above mentioned relation will be of the second order, if no fluents are contained in it.

Cor. From Py and Py' , and the points S and S' being given, can easily be deduced geometrically the direction of the tangent and the lines Sy , Sy' , &c.; for divide the line SS' in r , so that $Py \pm Py' : SS' :: Py : Sr$, and through r draw the line Pr , the perpendicular to Pr through P will be the tangent yPy' ; to this line the perpendiculars from S and S' will be the lines Sy and $S'y'$ required.

Cor. From the fluent of the above-mentioned fluxional equation may be deduced the velocity V in terms of D and D' ; and from the fluent of $\frac{D \times \dot{D}}{P_y \times V}$, which is a function of D multiplied into \dot{D} , may be deduced the time.

3. If the plane in which the body (P) moves, and all the forces $f', f'', f''', \&c.$ tending to points $M', M'', M''', \&c.$ not situated in the same plane (except one f tending to a given point M) be given, then the force tending to that point can be found, and the curve described. Resolve all the forces tending to the points $M, M', M'', M''', \&c.$ into two others; one $MS, M'S, M''S, M'''S, \&c.$ perpendicular to the plane in which the body moves, and the other $SP, S'P, S''P, S'''P, \&c.$ in the plane; then will $f \times \frac{MS}{MP} \pm f' \times \frac{M'S'}{M'P} \pm f'' \times \frac{M''S''}{M''P} \pm \&c. = 0$, from which equation f the force tending to the point M may be found; then, from the preceding proposition find the curve, which a body agitated by forces $f \times \frac{SP}{MP}, f' \times \frac{S'P}{M'P}, f'' \times \frac{S''P}{M''P}, \&c.$ tending to the points $S, S', S'', \&c.$ describes, and it will be the curve required.

4. If the body moves in a curve of double curvature, and the forces $f, f', f'', \&c.$ tending to all the centers $M, M', M'', M''', \&c.$ be given; from the fluent of the fluxional quantity $(f \times \frac{Py}{MP} \pm f' \times \frac{Py'}{M'P} \pm f'' \times \frac{Py''}{M''P} \pm f''' \times \frac{Py'''}{M'''P} \pm \&c.) \times \dot{A}$ (A denoting the same quantity as before) $= f \times \dot{M}P \pm f' \times \dot{M}'P \pm f'' \times \dot{M}''P \pm f''' \times \dot{M}'''P \pm \&c. = f \times \dot{D} \pm f' \times \dot{D}' \pm f'' \times \dot{D}'' \pm f''' \times \dot{D}''' \pm \&c. = \dot{Z} = -v\dot{v}$ ($f, f', f'', f''', \&c.$ being given functions of $D, D', D'', \&c.$ respectively) can be deduced the square of the velocity $= -2z$, which will be a function of

D, D', D'', D''', D''', &c., and consequently a function of D, D', D'', easily to be derived: substitute this function $-2z$ for v^2 in the two following equations $\frac{v^2}{R'} = F'$ and $\frac{v^2}{R''} = F''$, where R' and R'' denote the radii of curvature in two different planes of which the tangent above mentioned in Prob. 4. art. 4. is their intersection, and F' and F'' the sum of the forces in lines perpendicular to the tangent, and in the respective planes: from these forces, calculated in terms of the distances from three given points D, D', and D''; or in terms of two abscissæ and one ordinate, and from the radii R' and R'' may be deduced two fluxional equations of the second order, expressing the relation between three distances D, D', and D'', &c. which may always be reduced to one fluxional equation of the fourth order expressing the relation between one absciss and its correspondent ordinates, or the distances from two given points.

5. The general fluxional equation expressing the relation between the distances from two given points will be of the fourth order, if no fluents are contained in it; for it admits of four different quantities to be assumed at will, or according to the conditions of the problem.

6. If some points, to which the forces tend, are situated at an infinite distance; that is, some forces always act parallel to themselves; from the given forces acting either to given points, or in parallel directions, by the equation $f \times \dot{D} \pm f' \times \dot{D}' \pm f'' \times \dot{D}'' \pm \&c. = -v\dot{v}$ can be deduced the square of the velocity at a point P in terms of the distances from two given points, or of an absciss and ordinate; if the centers, &c. and parallel forces are all situated in the same plane: or in terms of the distances from three points, or two abscissæ and an ordi-

nate, if situated in different planes; from the centers, &c. and forces given, find the sum F of the forces in any direction (PL) (the direction of the tangent excepted) acting on the body at the point P , and the chord of curvature C of the curve at the same point and in the same direction; in the equation $v^2 = \frac{1}{2} F \times C$ for v^2 substitute the value found before, and there results an equation expressing the relation between the distances from two points, or an absciss and ordinate, &c. if the forces act in the same plane: but if the forces act in different planes, find the sum F and F' of the forces at the point P in directions which are not both situated in one plane with the tangent and each other; and also the chords C and C' of curvature in those directions in terms of the distances from three points, or two abscissæ and one ordinate, &c. In the equations $v^2 = \frac{1}{2} F \times C$ and $v^2 = \frac{1}{2} F' \times C'$ for v^2 substitute its value found from the principles before given; and there result two fluxional equations of the second order expressing the relation between the distances from three points, or two abscissæ and an ordinate, &c.

P R O P. VIII.

Fig. 9. Let a body move in a curve Pp , &c. and be acted on at P' by a force f (which is as any function of the distance SP') tending to S ; let the velocities at P and p be represented by the lines YP and yp in the direction of the tangents to the points P and p ; resolve these forces YP and yp into two others Yk and kP , and yl and lp , of which one kY and yl is parallel to the line SL ; the other kP and lp is parallel to MP : let a body fall in the right line LS , and the force acting on the body at M' be to the force acting on the body moving in the curve at $P' :: SM' : SP'$, and $P'M'$, PM and

and pm be perpendicular to SL ; then, if the velocity of the body falling in the right line SL at the point M be kY , the velocity of the body at the point m acted on by the above mentioned forces will be yl .

This is easily demonstrated from the resolution of forces.

2. Through S draw SN parallel to PM or pm , &c., and assume in the line (SN) $SP = PM$ and $Sp = pm$, and let the force at P' in the line SN and distance $= M'P'$: the force of the body moving in the curve at the distance $P'S : P'M' : SP'$; then if the velocity at the distance $SP = PM$ be Pk , the velocity at the distance $Sp = pm$ will be pl .

Cor. The force in the direction of the line SL vanishes in the point where a perpendicular SN to the line SL passing through the point S cuts the curve, and consequently the velocity in the direction of SL in that point is the greatest or least, &c.; but if the tangent of the curve be perpendicular in any point to LS , then the velocity in the direction LS is nothing: the same may be applied to the velocity in any other direction.

Ex. Fig. 10. Let a body move in the circumference of a circle SPA , of which the center of force is a point S in the circumference; it is known, that the force in the direction and at the distance SP is as SP^{-5} ; but the force in the direction SP is by the hypothesis to the force in the direction $(SA) :: SP : SM$, if PM be perpendicular to SM , and consequently the force in the direction (SA) is as $SM \times SP^{-6}$; but if AS be a diameter, $AS \times SM = SP^2$; therefore $SM \times SP^{-6} = SM \times AS^{-3} \times SM^{-3} = \frac{SM^{-2}}{AS^3}$; and the diameter AS being given, the force in the line

SA varies as SM^{-2} , that is, inversely as the square of the distance: if the force varies as $SM^{-2} = r^{-2}$, then $v\dot{v}$ will vary as

$-\frac{x}{x^2}$, where v denotes the velocity; and v^2 will vary as $\frac{1}{x} - \frac{1}{SA}$, which agrees with the square of the velocity deduced from the preceding principles; for $v = PY$ the velocity at P is inverfely as the perpendicular $SY = SM$ let fall from the center of force on the tangent; but $SA^2 : 2SP \times PA ::$ velocity PY as $\frac{1}{SY} = \frac{1}{SM}$: P the velocity at M ; whence Pl^2 (the square of the velocity at M) $= \frac{4SP^2 \times PA^2}{SA^4} \times PY^2$ which varies as $\frac{4SP^2 \times PA^2}{SA^4} \times \frac{1}{SM^2} = \frac{4PA^2}{SA^3 \times SM} = \frac{4SA^2 - 4SA \times x}{SA^3 \times x}$, and consequently as $\frac{1}{x} - \frac{1}{SA}$ the fame as above.

2. Fig. 9. If any number of forces act on a body at P in any given directions parallel, or tending to given points; resolve all the forces into two others; one in a given direction SM , and the other in a direction PM perpendicular to it, of which let F be the fum of the forces resulting in the direction MmS , and f the fum of the forces resulting in the direction PM ; resolve the velocity V of the body at P , which is in the the direction of the tangent PY , into two others V' and V'' , one in the direction parallel to the line SM , and the other perpendicular to it: in the same manner resolve the velocity v of the body at p , which is in the direction of the tangent py , into two others v' and v'' , one in the direction parallel to the line SM , and the other perpendicular to it: then if the velocity of the body moving in the right line SM at M be V' , and it is constantly acted on by a force $= F$, the velocity of the body at m will be v' : and if the body move from P in a direction perpendicular to SM with a velocity as V'' , and be always acted on by a force f , the velocity at the distance $PM - pm$ will be v'' .

Cor.

Cor. From the forces given and the velocities in the above mentioned directions at the point P, can be deduced the velocities in the same directions at the point p , and consequently the tangent to the curve at the point p .

P R O P. IX.

1. Let the resistance of a body, moving in a right line, be as any function V of the velocity v ; then will $t = \frac{\dot{v}}{V}$, $\dot{x} = \frac{-v\dot{v}}{V}$; where t , \dot{v} , and \dot{x} , denote the increments of time, velocity, and space; their fluents properly corrected will give the time and space in terms of the velocity.

2. Let a body move in a right line, and be acted on by an accelerating force in that line, which varies as any function X of the distance x from a given point; and resisted by a force which is as any function V of the velocity v into its density X' , which varies also as a function of x and v ; then will $(X + aVX')$ $\dot{x} = -v\dot{v}$, from its fluent x can be found in terms of v , or v in terms of x ; and thence $t = \frac{\dot{v}}{X + aVX'}$, of which the fluent properly corrected gives the time.

Ex. 1. Let $V = v^2$ and X' a function of x ; that is, let the resistance be as the square of the velocity and density, whence $(X + av^2X')$ $\dot{x} = -v\dot{v}$, of which equation the fluential will be

$$e^{\int 2aX'\dot{x}} X \frac{v^2}{2} = - \int e^{\int 2aX'\dot{x}} X X \dot{x} + A, \text{ and } t =$$

$$\int \frac{\dot{x}}{\sqrt{-\left(e^{-\int 2aX'\dot{x}} \times \left(\int e^{\int 2aX'\dot{x}} X X \dot{x} + A\right)\right)}} + B, \text{ where } A \text{ and } B \text{ are}$$

invariable quantities to be assumed according to the conditions of the problem.

1.2. Let $e^x = X'$ and $X = b$, which is supposed to correspond nearly to the state of our atmosphere, then will $v^2 = -2 \times e^{-\int 2aX\dot{x}} \times \int e^{\int 2aX\dot{x}} \times X\dot{x} = -2 \times e^{-\int 2ae^{x\dot{x}}} \int e^{\int 2ae^{x\dot{x}}} \times b\dot{x} = -2e^{-2ae^x - b} \times \left(\int e^{2ae^{x\dot{x} + b}} \times b\dot{x} + A \right)$, e being the number, whose hyperbolic log. is 1, and b and A quantities to be assumed according to the conditions of the problem.

1.3. Let $X = X'$, and it becomes $X\dot{x} = \frac{-v\dot{v}}{1+aV}$, and $t = \frac{-\dot{v}}{X(1+aV)}$.

2. Let X be an homogeneous function of one dimension of x , that is, $= ax$, and V a similar function of n dimensions of v , that is $= bv^n$, and X' a similar function of r dimensions of x and v , and $n+r=1$; then by substituting zx and its fluxion for v and its fluxion, can be found the fluent of the fluxional equation $(X+aVX')\dot{x} = -v\dot{v}$, and consequently the velocity and time by the quadrature of curves in terms of the space; and in like manner of many other cases.

3. Fig. 4. Let a body moving in a given curve be acted on at any point P by a force f tending to a given point S , and resisted by a medium proportional to V a function of its velocity multiplied into its density X' a function of the distance $SP = D$; to find its velocity, time, and distance from the given point S in terms of each other. Let $F = f \times \frac{PY}{SP}$ the force in the direction of the tangent PY , and consequently $(F + VX')\dot{A} = -v\dot{v}$, and $v^2 = \frac{1}{2} C \times f$, where \dot{A} is the increment of the arc, and C the chord of curvature in the direction SP ; but since the curve is given, the chord of curvature may be deduced from the distance, &c. and the increment \dot{A} of the arc from a function of the distance multiplied into the increment

of the distance; then, if f or v be a given function of the distance, the other may be deduced from it, and consequently $-v\dot{v} = \phi : (D) \times \dot{D}$ will be a given function of the distance D multiplied into \dot{D} , whence we have $\phi : (D) \times \dot{D} = \dot{D} (f \times \frac{PY}{SP} + X'V)$ divide by \dot{D} , and there results an algebraical equation, from which $V \times X'$ may be found.

If neither v nor f be given, reduce the two equations $(f \times \frac{PY}{SP} + VX') \dot{A} = -v\dot{v}$ and $v^2 = \frac{1}{2} Cf$ into one, so as to exterminate either f or v and its fluxions, and there results an equation expressing the relation between the other v or f and D and their fluxions: from the velocity given in terms of D may be deduced the time from the equation $t = \frac{\dot{A}}{v}$.

3.2. If the body be acted on by forces tending to more points $S, S', S'', S''', \&c.$ in the same plane; resolve each of the forces into two; one in the direction of the tangent, and the other perpendicular to it; let the sum of the forces in the direction of the tangent be F ; and in the direction perpendicular to it be F' ; and $2R$ the diameter of curvature at the point P , which will be given in terms of the distances from two points, or of an absciss and ordinate, and their fluxions, $\&c.$: assume the two equations before given $(F + X'V) \dot{A} = -v\dot{v}$ and $v^2 = F'R$, and since \dot{A} is always given in terms of D and \dot{D} , if F and F' be given in terms of $D, D', \&c.$ the value of $V \times X'$ may be acquired by a simple algebraical equation: but if F and F' be not given, and consequently v not given, but V a given function of v , and X' a given function of the above mentioned distances; then substitute for v its value $\sqrt{(F'R)}$ in the function V , and the fluxion of $\frac{1}{2} F'R$ for $v\dot{v}$, and there will result

an equation involving D and F' and their fluxions, and F ; but if the forces tending to all the points but one are given in terms of the distance D , or absciss or ordinate of the curve, and their fluxions; then from F' can be found F , and, *vice versa*, from F can be found F' , and consequently there results a fluxional equation expressing the relation between F or F' and the distance D or D' , &c. or absciss or ordinate, and their fluxions.

From F and F' , and consequently v being found in terms of D , D' , &c. can be deduced $t = \frac{v}{F}$.

The same method may be applied, if some forces tend to an infinite distance, that is, act parallel to themselves, and others tend to given points.

Ex. Let the accelerating force be directly as the arc $= x$, and the resistance uniform $= a$; then will $(x - a) \dot{x} = -v\dot{v}$, and consequently $x^2 - 2ax + B = -v^2$; let A be the arc, where the velocity $= 0$; then will the equation $A^2 - 2aA - x^2 + 2ax = v^2$, and the increment of the time $t = \frac{\dot{x}}{v} = \frac{\dot{x}}{\sqrt{(A^2 - 2aA - x^2 + 2ax)}}$,

whose integral is $\frac{1}{A-a} \times$ arc of a circle, of which the radius is $A - a$ and cos. $= x - a$, where A is the distance of the point from which the body begins to fall, and the lowest point of the curve; and the accelerating force $x - a$ is as the distance from a point (a) of a curve, of which the distance from the lowest is a .

Cor. The times of the body falling from any point of the curve to a will be equal.

Cor. The body on this hypothesis will either rest at the point a , or at the lowest point, or any point between $+a$ and

- a ; for it may rest at any point, where the resisting force is always equal or greater than the accelerating force.

Cor. Let n be the number of vibrations, then the distance of the arc, to which it will ascend from the lowest point at n vibrations, will be $A - 2na$; if $A - 2na$ be not greater than $2a$, it will never pass the lowest point.

Philosophical enquiries require some corrections, which do not enter into mathematical calculus; for example, in some cases the calculus changes the quantities from negative to affirmative, &c. when from philosophical considerations they are not changed; and, *vice versa*, they may be changed to affirmative, &c. on philosophical considerations, when they are not changed from the calculus: and also a body may stop, &c. from philosophical considerations, as in the preceding example, when it does not follow from the algebraical calculus, &c. It is further to be observed, that resistances are always to be taken affirmatively.

Ex. 2. Let the accelerating force be as the arc, that is, the distance from the lowest point, and the resistance as the velocity; then will the fluxional equation $(F - V) \dot{A} = -v\dot{v}$ be $(ax - v) \dot{x} = -v\dot{v}$, which is an homogeneous equation of the first order: write in it zx for v , and its fluxion for \dot{v} , and there results the equation $(ax - zx) \times \dot{x} = -zx^2\dot{z} - z^2x\dot{x}$, whence $(a - z) \dot{x} = -zx\dot{z} - z^2\dot{x}$ and $\frac{\dot{x}}{x} = \frac{-z\dot{z}}{a - z + z^2}$, and thence $\log. x = -\frac{1}{2} \log. (a - z + z^2) (W) - \frac{2}{4a - 1} \times \text{cir. arc}$, whose radius is $\frac{\sqrt{4a - 1}}{2}$ and tangent $(z - \frac{1}{2}) + B$, whence can be found $v = xz$, and from curvilinear areas $t = \frac{\dot{x}}{v}$.

If $4a$ is less than 1, then it becomes $\log. x = W - \frac{1}{4\sqrt{\frac{1}{4}-a}} \times$
 $\log. \frac{x - \sqrt{\frac{1}{4}-a-\frac{1}{2}}}{x + \sqrt{\frac{1}{4}-a-\frac{1}{2}}} + B$; where B is an invariable quantity to be
 assumed according to the conditions of the problem.

Cor. If the force be directly as the distance, or as the arc of the curve from the body to the lowest point, and the resistance as the velocity; then will the velocity in one arc be to the velocity in the corresponding point of another arc, as the arcs to be described; and consequently the times equal.

4. If the body is acted on by forces tending to points $S, S', S'',$ &c. situated in different planes, then let F be the sum of the forces in the direction of the tangent at the point P ; F' and F'' the sum of the forces acting on the body in two different directions at the same point, which are not situated in the same plane with the tangent and each other; from the three equations $(F + X/V) \dot{A} = -v\dot{v}$ and $\frac{v^2}{C} = \frac{1}{2}F'$ and $\frac{v^2}{C'} = \frac{1}{2}F''$, in which the same letters denote the same quantities as before, and C and C' denotes the chords of curvature in the same directions as the forces F' and F'' , which from the curve being given can be found at any point; and if F' or F'' is given in terms of the distance from a given point, or an absciss or ordinate, &c. the velocity v can be found in terms of the same, and X/V by a simple algebraical equation: if F' is not given, and V is a given function of v , substitute in V for v its value $\sqrt{(\frac{1}{2}C \times F')}$, and there results an equation expressing the relation between F (which can be deduced from F' or F'') and the distance of the body from some given point, or the abscissæ and ordinates of the curve required, and their fluxions.

If

If some of the forces act in parallel directions; the forces, velocities, &c. may be found by the same method.

P R O P. X.

Fig. 11. Let a body be projected in a direction HL with a given velocity, and be acted on by a force in a direction parallel to AP = x , which varies as X a function of x ; and also by another force in a direction parallel to MP = y , that is, perpendicular to AP, which force varies as Y a function of y ; and let it move in a medium, of which the resistance is proportional to the velocity; to find the curve described.

Find the fluent of $(X + av) \dot{x} = -v\dot{v}$, which corrected according to the conditions of the problem (*viz.* so that v at the point H may be to the velocity of projection :: Hc : Hb, where bc is drawn perpendicular to AP) suppose $v = X'$; find the fluent of $\frac{\dot{x}}{X'}$, which corrected so as to become = 0, when $x = AH$, let be X'' . In the same manner find the fluent of $(Y + a'v') \dot{y} = -v'\dot{v}'$, which corrected, so that v' at the point H may be to the velocity of projection :: cb : Hb, suppose $v' = Y'$; find the fluent of $\frac{\dot{y}}{Y'}$, which corrected so as to become = 0, when $PM = 0$, let be = Y'' ; assume $X'' = Y''$, and thence from x find y : take AP = x and PM = y , and M will be a point in the curve, which a body projected in the line HL describes; and if Mm in the direction parallel to HAP : mo perpendicular to it :: velocity v : velocity v' , then will Mo be a tangent to the curve in the point M.

2. If a body is acted on by forces tending to any given points S, S', S'', &c. which vary as given functions of their distances from the body, and resisted by a force which varies according to a given function V of the velocity (v) into its density X',

where X' varies according to some function of the distances from the given points, &c. ; to find the curve described.

1. From the distances of the body from two given points, or the absciss and ordinate of the curve described, and their fluxions, &c. find the forces acting in the direction of the tangent to the curve, and in some other direction, which suppose F and F' ; and also the chord of curvature in the above mentioned direction, which let be C ; then from the equations $(F + V \times X')$ $\dot{A} = -v\dot{v}$ and $v^2 = \frac{1}{2} C \times F$ reduced into one by writing for v its value in the function V , and for $v\dot{v}$ its value deduced from the equation $v^2 = \frac{1}{2} C \times F$, and for \dot{A} (the fluxion of the arc) its value deduced from the distances, &c. will result an equation expressing the relation between the distances from two given points to the curve, or its absciss and ordinates, and their fluxions.

3. If the forces are not all situated in the same plane, then from the before given equation $(F + V \times X') \dot{A} = -v\dot{v}$, and the two others $v^2 = \frac{1}{2} C \times F'$ and $v^2 = \frac{1}{2} C' F''$, where F denotes the force in the direction of the tangent, and F' and F'' are the forces in different directions, which both are not situated in the same plane with each other and the tangent, and in which directions the chords of curvature are respectively C and C' ; since the quantities F , F' , and F'' ; C and C' and \dot{A} (as proved before) can all be expressed in terms of the distances from three given points, or from two abscissæ and one ordinate, and their respective fluxions ; may be deduced two fluxional equations expressing the relation between the distances from three given points, or two abscissæ and an ordinate, &c.

The same principles may be applied to cases, in which some of the forces act in parallel directions.

Centripetal Forces.

On moveable Centers.

P R O P. XI.

1. Given the respective places of (n) bodies $S, S', S'', S''',$ &c. in the curves $A, A', A'', A''',$ &c. at the same time, and in the same plane, and the forces of all the bodies acting on S , except two, S' and S'' ; to find the forces of the two bodies S' and S'' on the body S .

This proposition may be resolved by the method given in Prop. 4. for to produce the same effect the same finite forces will be requisite, whether the centers of forces rest or move in given curves.

1.2. If the bodies $S, S', S'',$ &c. move in different planes, then all the forces acting on the body, except three, may be given, which may be acquired from the method given in the same proposition.

Hence it appears, that $2n$ forces may be requisite to be found from the conditions of the problem to determine all the bodies to move in their respective curves, when they are all situated in the same plane, and that $3 \times n$ forces may be requisite in different planes, &c. if the force of one body (S') on another (S'') does not at all depend on the force of the same body (S') on any other (S'''); and if the same can be prædicated of the rest, then $n \cdot \overline{n-3}$ forces of the above mentioned bodies in the same, or $n \cdot \overline{n-4}$ forces in different planes may be assumed at will.

3. If the velocities $v, v', v'',$ &c. at every point of the arcs $a, a', a'',$ &c. of the (n) above mentioned curves $A, A', A'',$ &c. be given in terms of their arcs, absciffæ, or ordinates, &c. and the places in which the bodies are situated at the fame time in the arcs $b, b', b'',$ &c. of fome other curves $B, B', B'',$ &c. find the correfponding velocities $V, V', V'',$ &c. at the fame time of the bodies in the curves $B, B', B'',$ &c.; then make $\frac{\dot{a}}{v} = \frac{\dot{a}'}{v'} = \frac{\dot{a}''}{v''} = \&c. = \frac{\dot{b}}{V}$, or which is equal to it = $\frac{\dot{b}'}{V'} = \frac{\dot{b}''}{V''} = \&c.$ From the fluents of the fluxional equations refulting properly corrected will be found the arcs $a, a', a'',$ &c. described by the bodies in the curves $A, A', A'',$ &c. in the fame time as the correfpondent arcs b, b', b'' &c.; and from thence, by the method given in the preceding cafe, may be deduced the forces.

The fame principles may be applied to bodies moving in refifting mediums.

P R O P. XII.

Given the law of the forces of two bodies acting on each other, to find the two curves by them described.

Fig. 12. Affume x and y for the abfcifs (AP) and ordinate (PM) of one curve, and z and u for the abfcifs (AP') and ordinate (P'M') of the other; where the abfciffæ AP and AP' begin from the fame point A, and are fituated in the fame line; then will the diftance ($D = M'M$) between the bodies = $\sqrt{z \pm x^2} + \sqrt{u \pm y^2}$; let the forces of the body placed at M on that at M', and of the body placed at M' on that at M vary as $\phi : (D) = F$, and $\phi' : (D) = F'$; and let $M\dot{p} = \dot{x}$ and $p\dot{m} = \dot{y}$; then will cofine of the angle mMM' to radius (1) be $\frac{y \pm u}{D} \times$

$\frac{y}{\sqrt{(x^2+y^2)}} \pm \frac{x \pm z}{D} \times \frac{\dot{x}}{\sqrt{(x^2+y^2)}} = c$; and consequently the force in the direction of the tangent Mm will be $c \times F$, whence $-v\dot{v} = c \times F \times \sqrt{x^2+y^2} (\dot{A})$ and $v^2 = \frac{1}{2} CF$, where C is the chord of curvature in the direction of the force $(F) = \sqrt{1-c^2} \times 2 \frac{(x^2+y^2)^{\frac{1}{2}}}{yx-\dot{x}y}$; and v the velocity of the body in the curve, whose absciss is x and ordinate y .

In the same manner let $\frac{x \pm z}{D} \times \frac{\dot{z}}{\sqrt{z^2+u^2}} \pm \frac{y \pm u}{D} \times \frac{\dot{u}}{\sqrt{z^2+u^2}} = c'$, the cosine of the angle made between the distance MM' and arc of the curve of which the absciss is z and ordinate u , and consequently $c' \times F'$ will be the force in the direction of its tangent, and therefore $-v'\dot{v}' = c' \times F' \times \sqrt{z^2+u^2} (\dot{A})$ and $v'^2 = \frac{1}{2} C'F'$, where C' is the chord of curvature in the direction of the force $(F') = \sqrt{1-c'^2} \times 2 \frac{(z^2+u^2)^{\frac{1}{2}}}{(uz-\dot{z}u)}$, and v' the velocity of the body in the curve whose absciss is z and ordinate u ; then, because the times of describing correspondent arcs in the two curves are equal, their increments will be equal, and consequently $t = \frac{\sqrt{x^2+y^2}}{v} = \frac{\sqrt{z^2+u^2}}{v'}$; and there are deduced five fluxional equations, containing six variable quantities v, v' ; x, y, z , and u , and their fluxions; reduce these equations, so that four of them ($v, v', \&c.$) may be exterminated, and there will result an equation expressing the relation between x and y the absciss and ordinate of one curve, or z and u the absciss and ordinate of the other curve, and their fluxions; the fluxional equation of which being found, and properly corrected, gives the equation to the curve.

The five equations are easily reduced to three by exterminating the quantities v and v' .

The fluxional equation resulting will most commonly be of the fifth order, as evidently appears from the nature of the problem.

2. The same principles may be applied to determine the curves, when the bodies move in mediums, of which the resistances are given: for example, suppose the resistances to vary as a function of the distance from a given point into a function of the velocity: to the forces in the directions of the tangents contained in the preceding case must be added or subtracted the given resistances for the forces in the directions of the tangents, and the remaining process will be the same as is before given.

If two bodies describe similar orbits round a common center, either quiescent or moving uniformly in a right line; the forces and velocities and resistances of the medium will be to each other in correspondent points as their respective distances from the center.

P R O P. XIII.

Given the forces acting on any bodies, and tending to points either moveable or quiescent, or in the direction of the tangents, &c.; to find the curve described by one of the bodies.

1. Assume x and y for the absciss and ordinate of the curve required, and from thence may be deduced the distances from any quiescent center of force, and consequently the force f in that direction; resolve it into two others, one in the direction of the tangent, and the other in a different one; for example, let it be in a direction perpendicular to the tangent, and from their fluxions \dot{x} and \dot{y} , and the force f may, by the method

before given, be deduced the forces in the two above mentioned directions; and in the same manner may be found from x , y , \dot{x} , and \dot{y} , the forces in the directions of the tangent and perpendicular to it, which follow from all the forces tending to given points, and acting on the body moving in the curve to be investigated. 2. If some of the centers of force move in given curves B, B', B'', &c. whose arcs let be denoted by B, B', &c. and their respective places at the same time are given; then from their respective places given and forces, and x and y , and \dot{x} and \dot{y} , can, as before, be deduced the forces in the direction of the tangent and its perpendicular to the curve required. 3. If other centers of forces move in given curves A, A', A'', &c. and the velocities are given at every point of the curves; let A, A', A'', &c. be the arcs of the curves A, A', A'', &c. and suppose v , v' , v'' , &c. their correspondent velocities; then, if the increments of the time be given, will $\frac{\dot{A}}{v} = \frac{\dot{A}'}{v'} = \frac{\dot{A}''}{v''} = \&c.$ but as the velocities are given at every point of the curves, v in the curve (A) will be given in terms of its absciss, ordinate, arc, &c. and consequently $\frac{\dot{A}}{v}$ in terms of the same quantities and their first fluxions; the same may be affirmed of the fluxions $\frac{\dot{A}'}{v'}$, $\frac{\dot{A}''}{v''}$, in the curves A', A'', &c.; hence, from the equation $\frac{\dot{A}}{v} = \frac{\dot{A}'}{v'}$, can be deduced the relation between the absciss or ordinate, &c. of the curve A and its correspondent absciss or ordinate, &c. of the curve A'; and so of the remaining curves; hence this case is reduced to the preceding; but it is necessary also, that the times of the bodies in the two cases should be the same, in order that the places may correspond, and consequently $\frac{\dot{A}}{v} = \frac{\dot{B}}{V}$, where V

denotes the velocity of the body at any point of the curve B from which equation can be deduced the correspondent abscissæ and ordinates, &c. of the curves B and A; and thence the two cases are reduced to the preceding, whence the correspondent forces in the directions of the tangent, and perpendicular to it, can be found as above. 4. If some (m) of the centers move in curves L, L', L'', &c. to be deduced from the laws of the forces being given which act on them; assume z and u , z' and u' , z'' and u'' , &c. for their respective abscissæ and correspondent ordinates; and from them and y and x , y' and x' , find the forces acting on the body moving in the curve required in the direction of the tangent, and perpendicular to it, as before; then add all the forces deduced which act perpendicular to the tangent and also all contained in the direction of the tangent together with the resisting force in the same direction, and let the sums resulting be respectively F and F' : by the same method find the sum of the forces which act on the bodies moving in L, L', L'', &c. in the directions of the tangents, and perpendiculars to them, which suppose S and s , S' and s' , S'' and s'' , &c.; then reduce the z ($m+1$) equations of the formulæ found above, viz. $v^2 = F \times \frac{x^2 + y^2^{1/2}}{yx - xy}$ and $-v\dot{v} = F'$

$$\sqrt{x^2 + y^2}; \quad v'^2 = s \times \frac{z^2 + u^2^{1/2}}{uz - zu} \quad \text{and} \quad -v'\dot{v}' = S \times \sqrt{z^2 + u^2}; \quad v''^2 =$$

$$s' \times \frac{z'^2 + u'^2^{1/2}}{u'z' - z'u'} \quad \text{and} \quad -v'' \times \dot{v}'' = S' \times \sqrt{z'^2 + u'^2}, \text{ \&c. ; where } v,$$

$$v', v'', v''', \text{ \&c. respectively denote the correspondent velocities of the bodies moving in the curves, whose abscissæ are } x, x', x'', \text{ \&c. ; and also the } (m+1) \text{ equations } \frac{\dot{B}}{v} = \frac{\sqrt{x^2 + y^2}}{v} =$$

$$\frac{\sqrt{z^2 + u^2}}{v'} = \frac{\sqrt{z'^2 + u'^2}}{v''} = \frac{\sqrt{z''^2 + u''^2}}{v'''} = \text{\&c. containing the } 3(m+1)$$

+ 1 variable quantities x and y , z and u , z' and u' , z'' and u'' , &c., v , v' , v'' , &c., and the variable quantity contained in B and V, into one, so that all the variable quantities except x and y and their fluxions may be exterminated, and there results an equation to the curve required expressing the relation between x and y its absciss and ordinate, and their fluxions.

5. If the forces are not situated in the same plane, assume X, x and y , for the two abscissæ and ordinates of the curve required; and Z, z and u ; Z', z' and u' ; Z'', z'' and u'' ; &c. for the two abscissæ and ordinates of the (m) curves L, L', L'', &c. respectively; and from the preceding method may be acquired the 3 ($m+1$) equations $v^2 = F \times C$, $v'^2 = F' \times C'$, and $-v\dot{v} = F'' \times \sqrt{X^2 + x^2 + y^2}$; $v'^2 = S \times C'$, $v'^2 = \sigma c'$ and $-v'\dot{v}' = s \times \sqrt{Z^2 + z^2 + u^2}$; $v''^2 = S' C'' = \sigma' c''$ and $-v''\dot{v}'' = s' \times \sqrt{Z'^2 + z'^2 + u'^2}$; $v'''^2 = S'' C''' = \sigma'' c'''$ and $-v''' \dot{v}''' = s'' \times \sqrt{Z''^2 + z''^2 + u''^2}$; &c.; in which v denotes the velocity in the required curve, and v' , v'' , v''' , &c. the correspondent velocities in the curves L, L', L'', &c.; and F, F', and F''; S, σ and s ; S', σ' and s' ; S'', σ'' and s'' ; &c. denote the forces acting on the respective bodies in two different planes and in the tangents, which planes cut each other in the tangents of the curves; and C and c , &c., C' and c' , &c., C'' and c'' , &c. the $\frac{1}{2}$ chords or radii of curvature in those two planes to the different curves in the directions of the forces; and also the

$$(m+1) \text{ equations before mentioned } \frac{\dot{B}}{V} = \frac{\sqrt{X^2 + x^2 + y^2}}{v} = \frac{\sqrt{Z^2 + z^2 + u^2}}{v'}$$

$$= \frac{\sqrt{Z'^2 + z'^2 + u'^2}}{v''} = \text{\&c.}; \text{ where } \sqrt{X^2 + x^2 + y^2}, \sqrt{Z^2 + z^2 + u^2},$$

&c. are the fluxions of the arcs of the required curve, and of the curves L, L', L'', &c. reduce these $4m+4$ equations containing

taining $4m + 5$ variable quantities into two, so that all the variable quantities except three, X , x , and y , and their fluxions may be exterminated; and there result the two equations required.

It may be observed, that when the resistance arising from the density of the medium and velocity (v) of the body varies as $X' \times v^2 + X$, where X and X' are as functions of the distances from given points, the resolution of the fluxional equations will generally be more easy, than when the resistance varies as other functions of the velocities.

If the force acts equally on the particles of the body and fluid, then the force by which a body descends in a medium is as the whole force X acting on the body at the given distance multiplied into a fraction whose numerator is the difference between the density of the body (D) and fluid (X') at that distance and denominator D , that is, as $X \times \frac{D - X'}{D}$.

Many cases might have been given, in which the fluxional equations could have been resolved; but in general their fluents can only be found by means of converging serieses.

